# ZX with Discards

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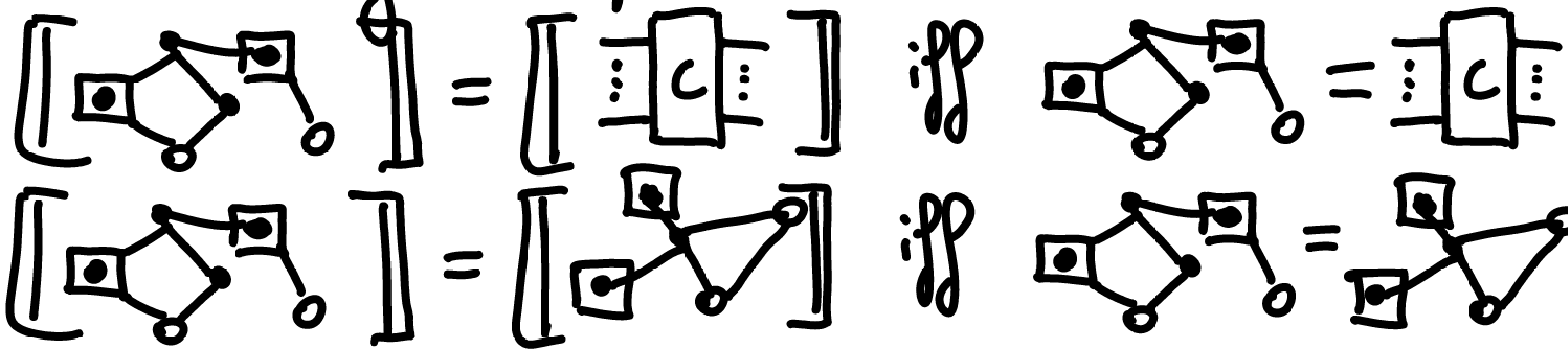
# Motivations

* Measurements with partial trace
* General construction from pure theory
* Preservation of completeness

$\Rightarrow$

$\Rightarrow$

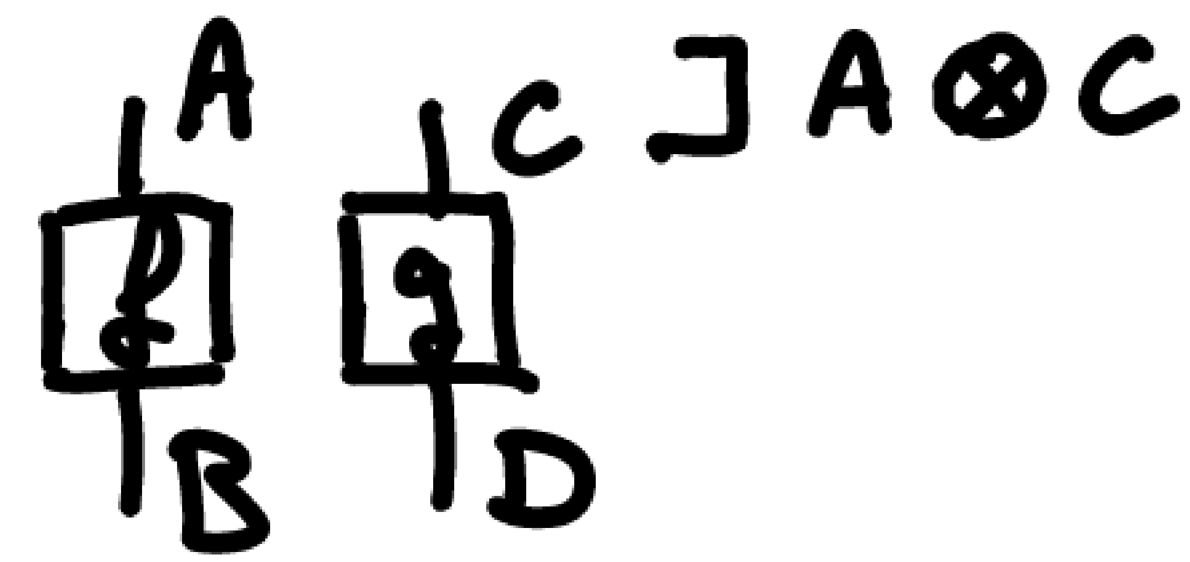


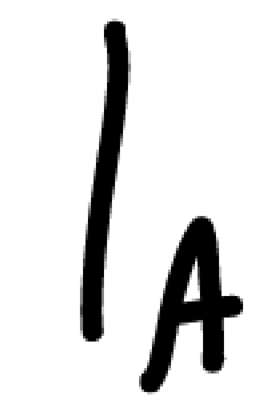

* [Huot, Staton' 18] : similar for CPTP maps

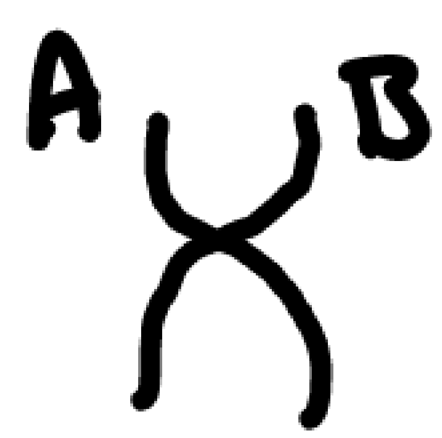
# †-Symmetric Monoidal Categories

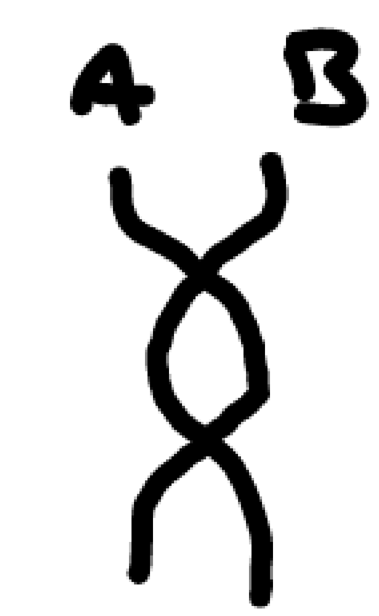
(with string diagrams)

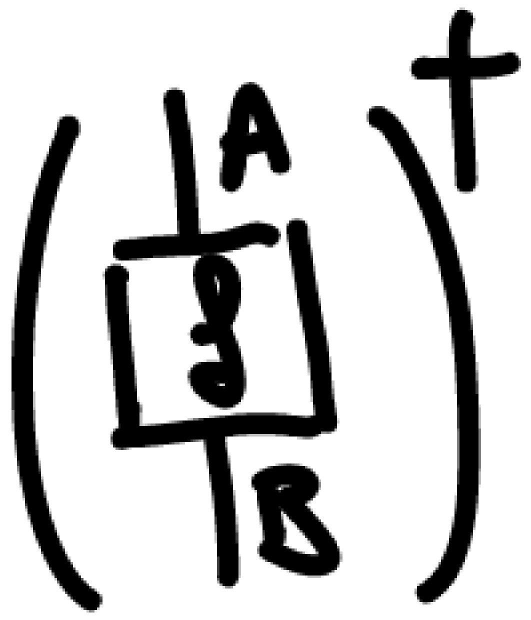
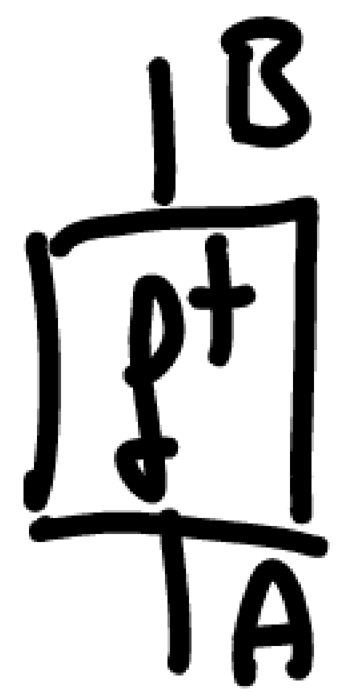
$g \circ f :=$ 

$f \otimes g :=$ 

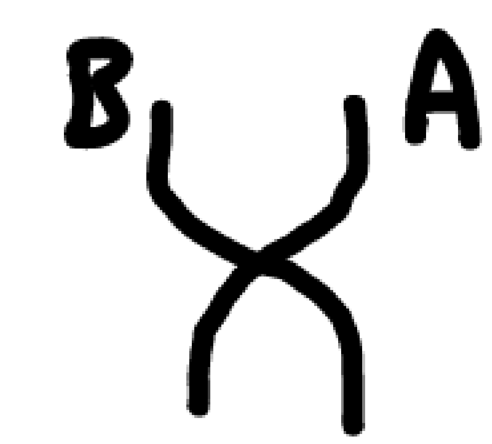
$id_A :$ 

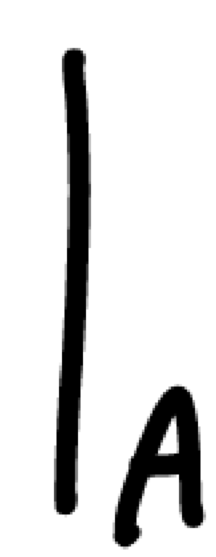
$\sigma_{A,B} :$ 

+ |coherence / naturality| axioms e.g. 

 $:=$ 

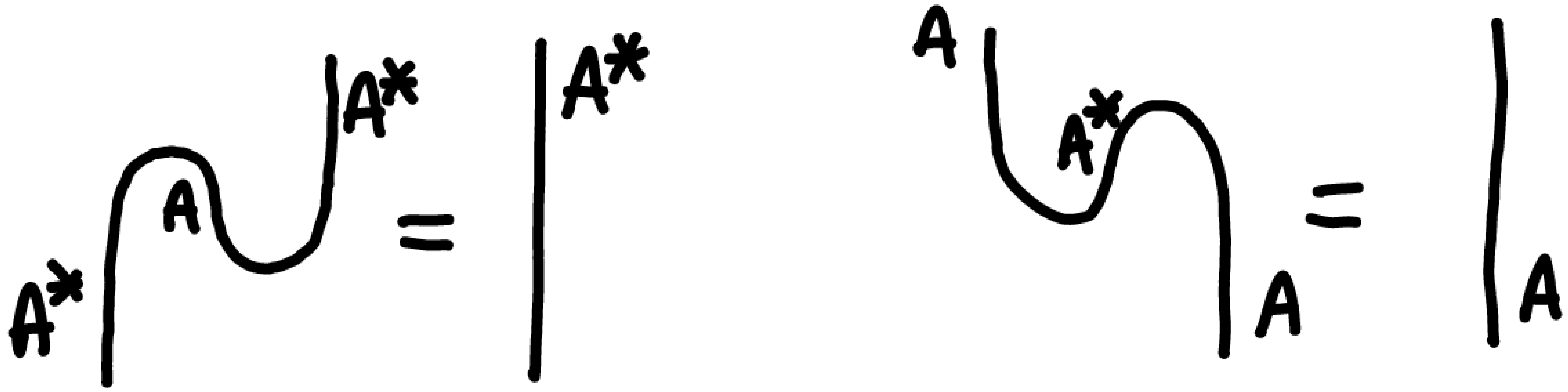
$f^{\dagger\dagger} = f$

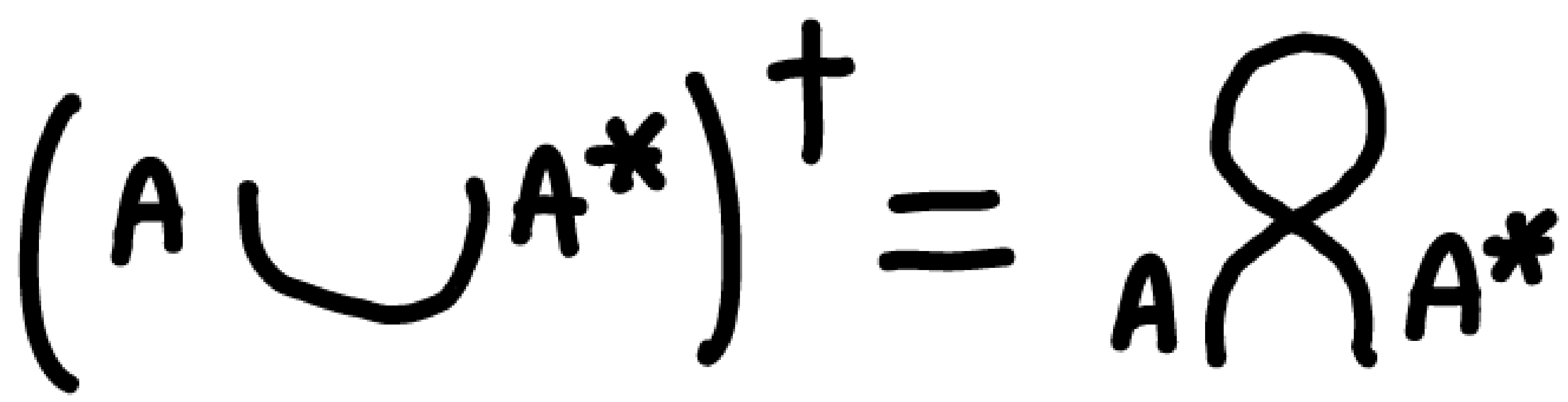
 $:=$ 

Def: isometry :  $=$ 

# Compact structure


with $A^*$ dual to $A$



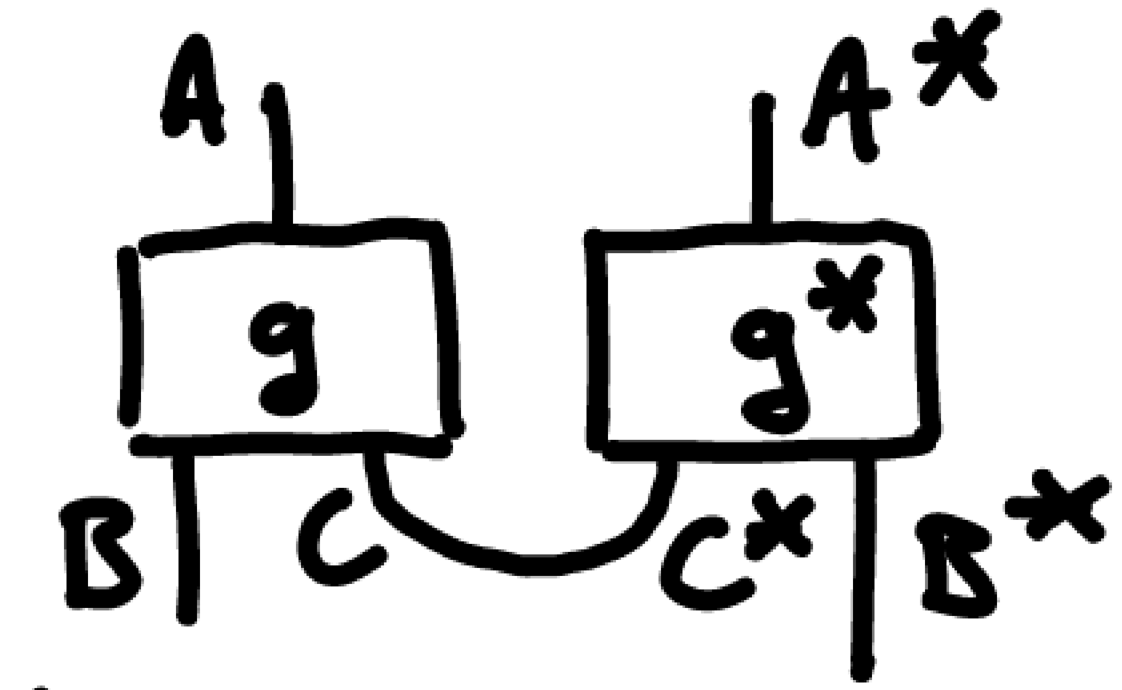



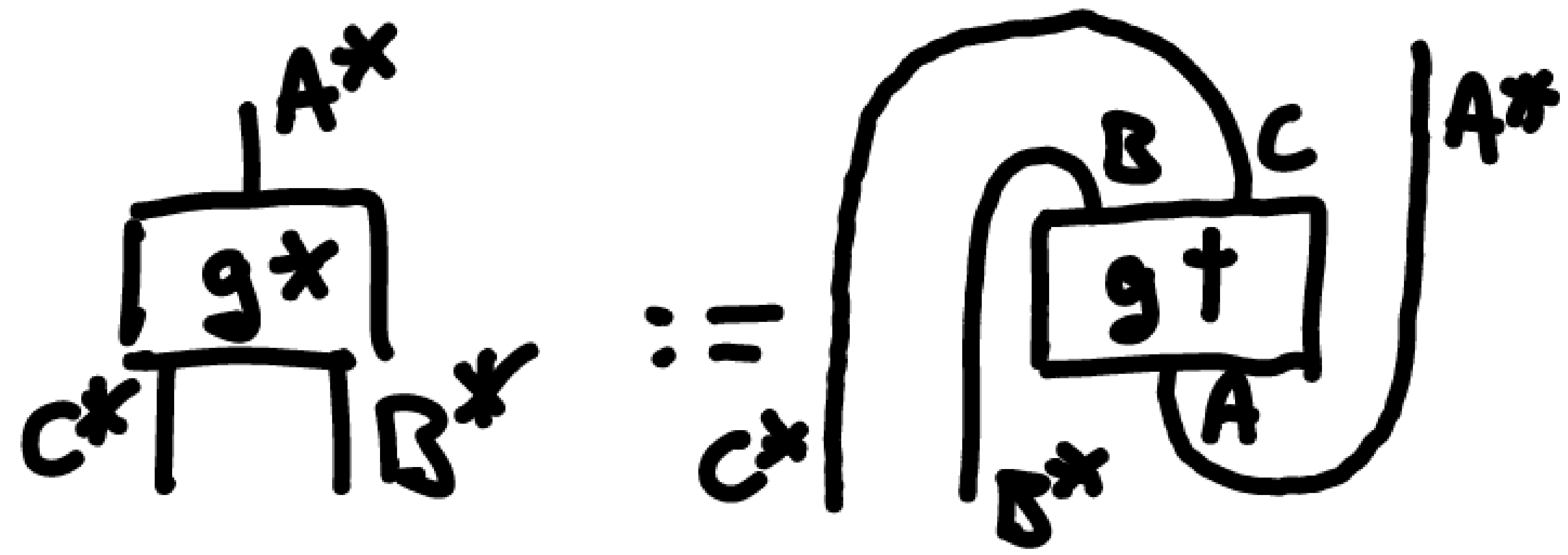

$\dagger$-compact prop: All of the above with a single generating object: $A^{\otimes n}$

Eg: ZX with $A := \mathbb{C}^2$

# From Pure to Mixed

* CPM construction:

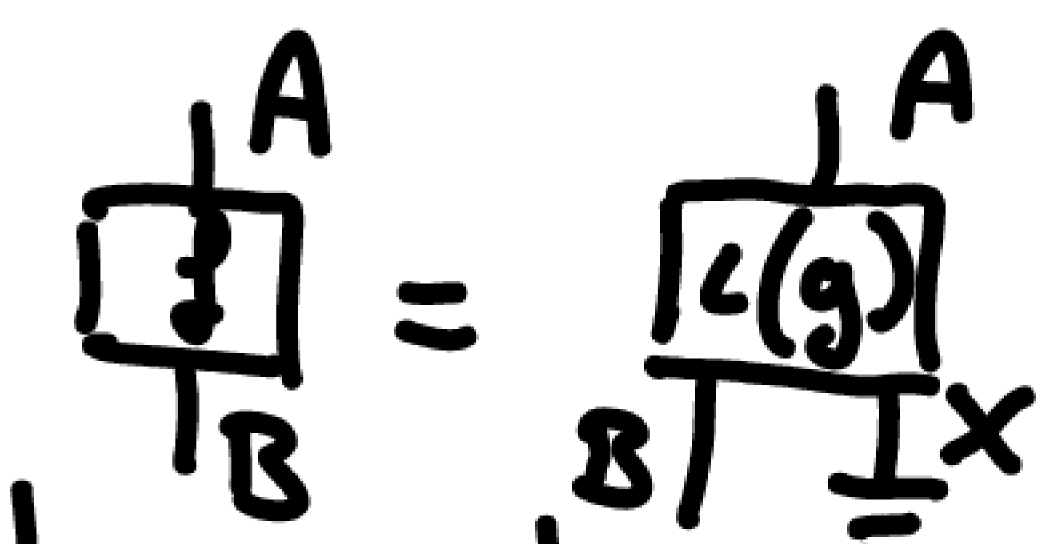
$f: A \to B \in CPM(C)$ iff $f =$ 

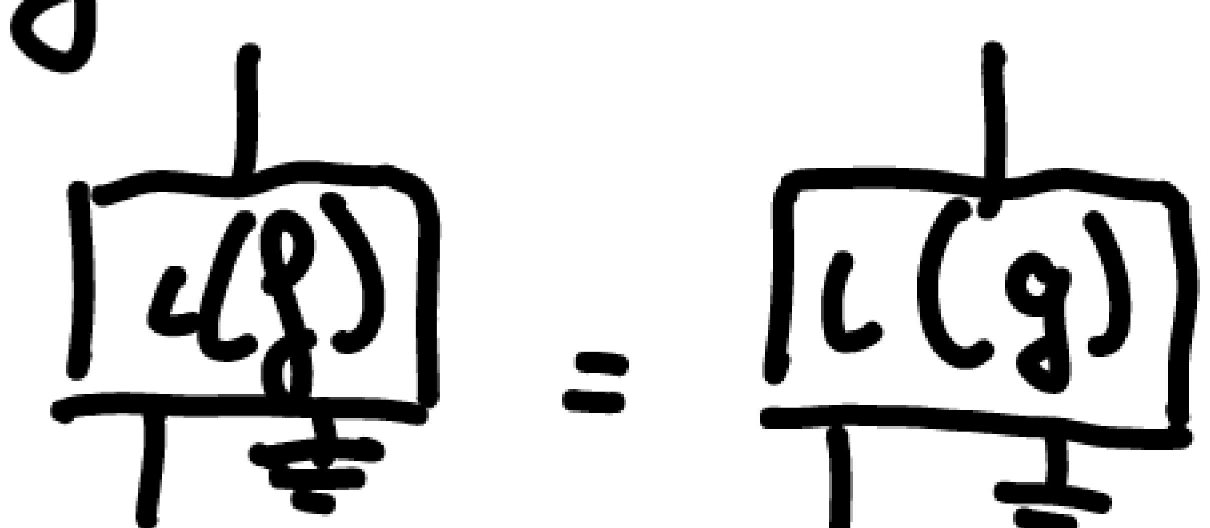

where 

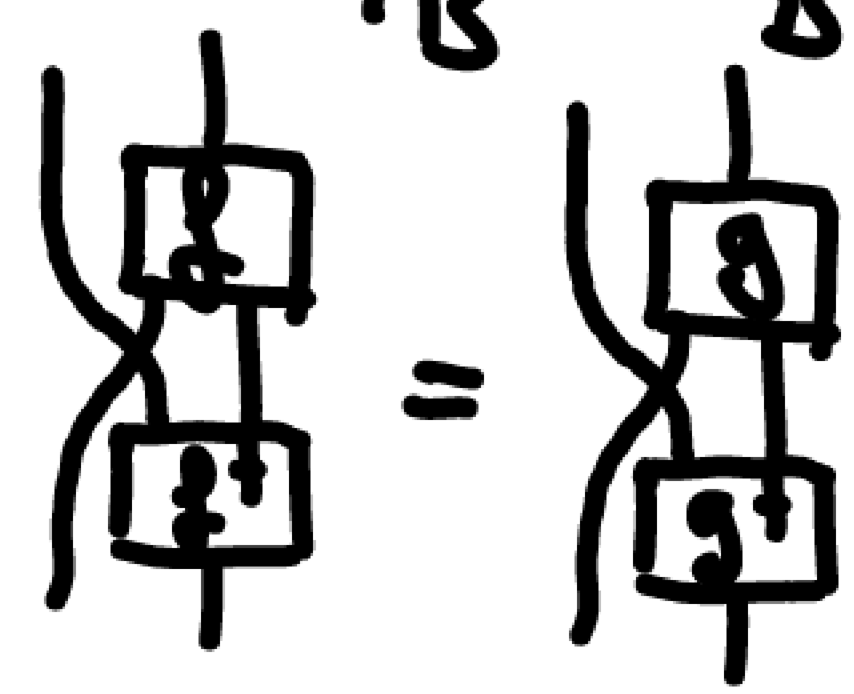

* Environment structure:

$\iota: C \hookrightarrow \bar{C}$ identity on objects

$\perp^A$ for all $A$, s.t. $\perp^A \perp^B \simeq \perp^{A \otimes B}$

$f: A \to B \in \bar{C}$ iff 

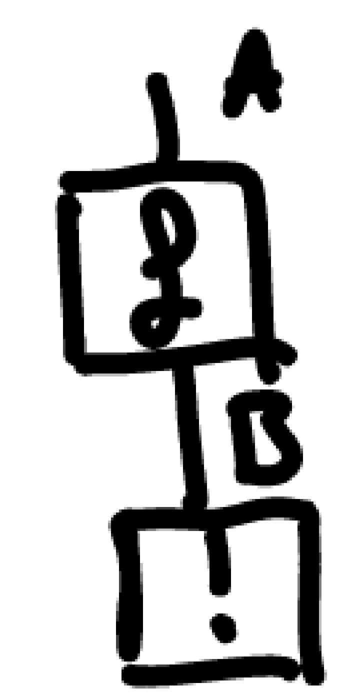
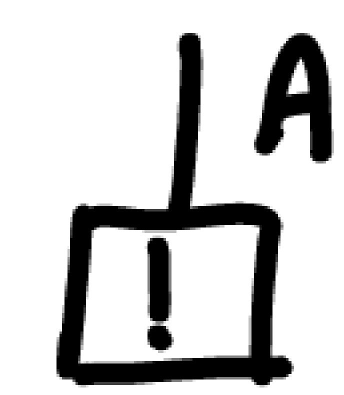




iff 

# Isometries & Affine Completion

* $\mathcal{C}_{iso}$ : subcategory of $\mathcal{C}$ that contains the isometries $f^\dagger \circ f = id$.  $\mathcal{C}_{iso} \hookrightarrow \mathcal{C}$

* $\mathcal{C}^!$ : affine completion of $\mathcal{C}$ :

$\forall A$ :  s.t. $\forall f$ :  $=$ 

$$\begin{tikzcd}
\mathcal{C}_{iso} \arrow[r] \arrow[d, hook] & \mathcal{C}_{iso}^! \\
\mathcal{C} &
\end{tikzcd}$$

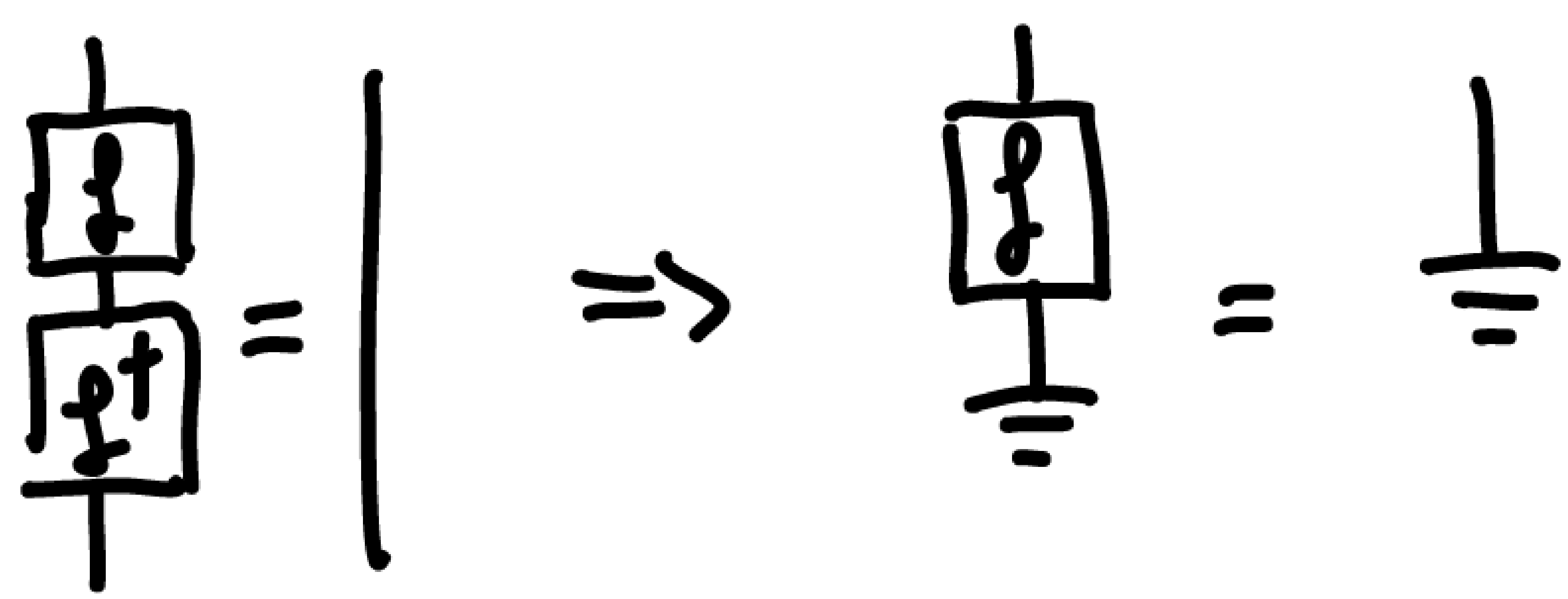
# Getting $C^{\text{⏚}}$

$$\begin{tikzcd}
C_{iso} \arrow[r] \arrow[d, hook] & C^{!}_{iso} \arrow[d] \\
C \arrow[r] & C^{\text{⏚}} \arrow[ul, phantom, "\ulcorner", very near start]
\end{tikzcd}
\qquad
\begin{tikzcd}
!_A \arrow[d, mapsto] \\
\text{⏚}_A
\end{tikzcd}$$

← pushout

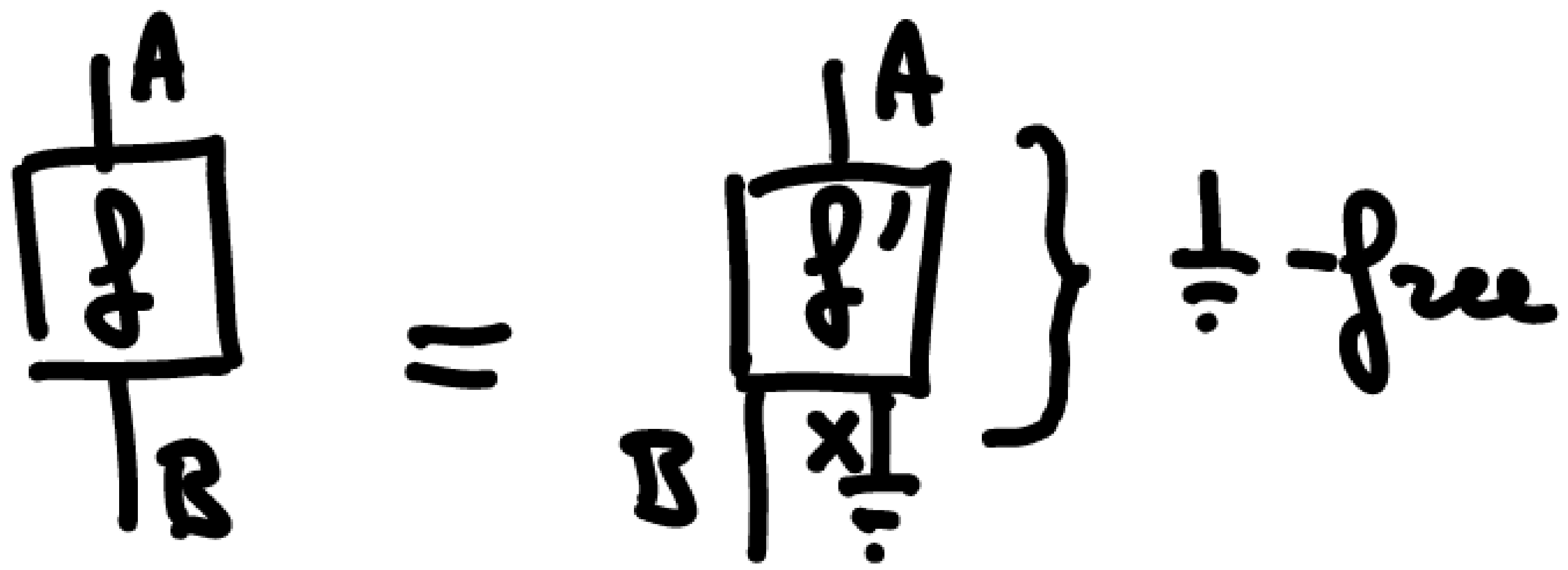
Universal construction

$C^{\text{⏚}}$ : $C$ + discards that delete isometries

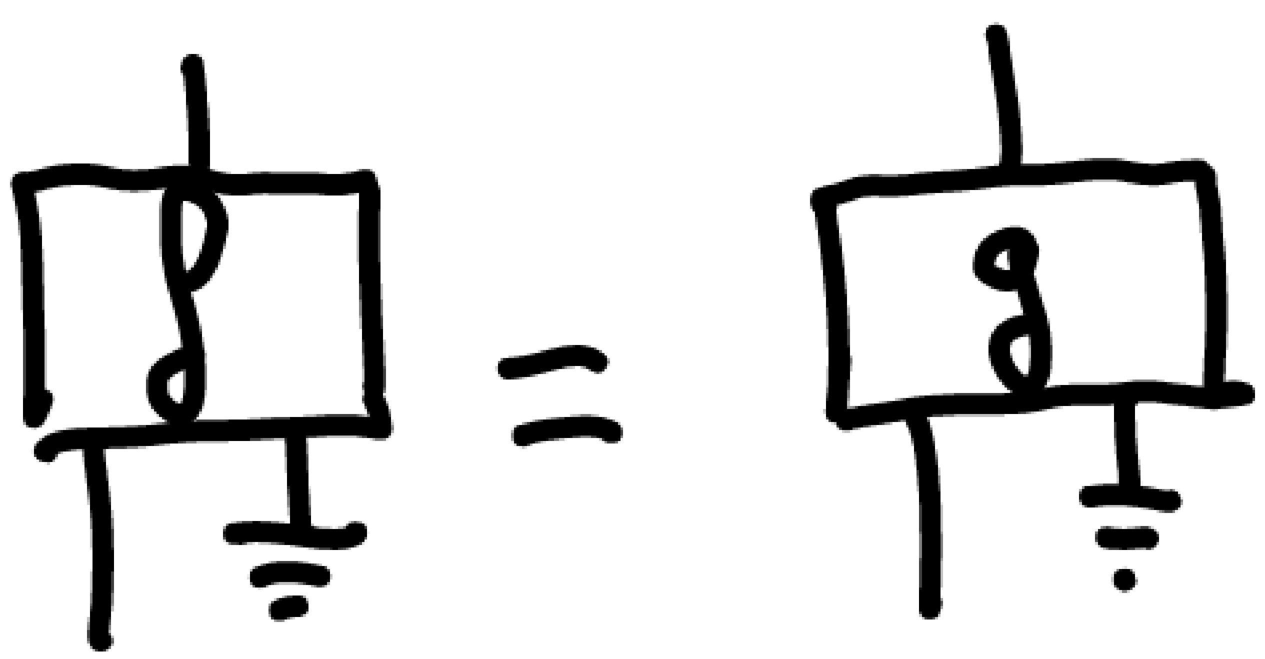


# Purification

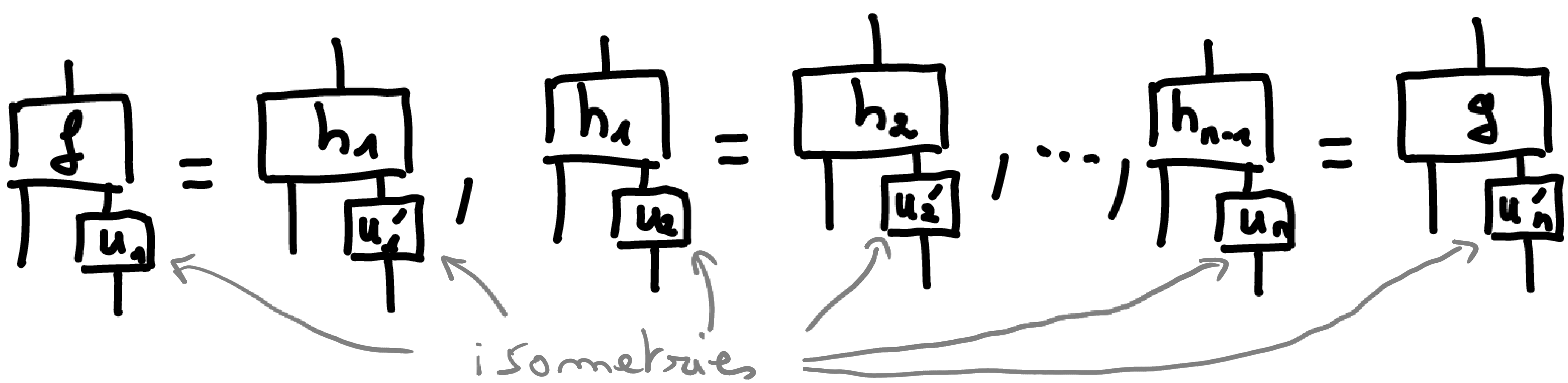
**Lem** :



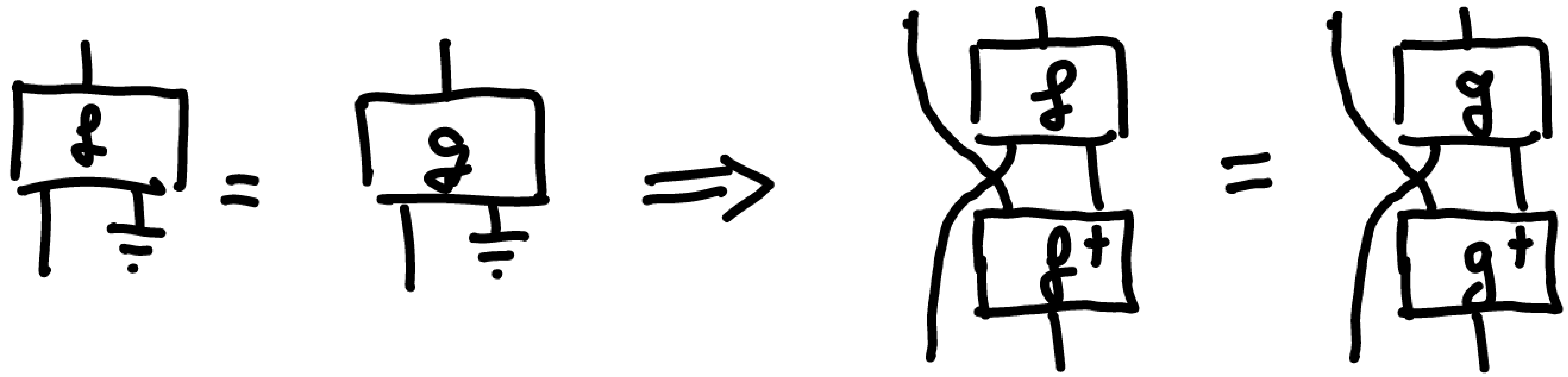

**Lem** :




$$\Longleftrightarrow$$



# Having Enough Isometries




**Def:** "C has enough isometries" if equivalence

**Lem:** C has enough isometries

$\Updownarrow$

$C^{\ddagger}$ is an environment structure for C

$\Updownarrow$

$C^{\ddagger} \simeq CPM(C)$

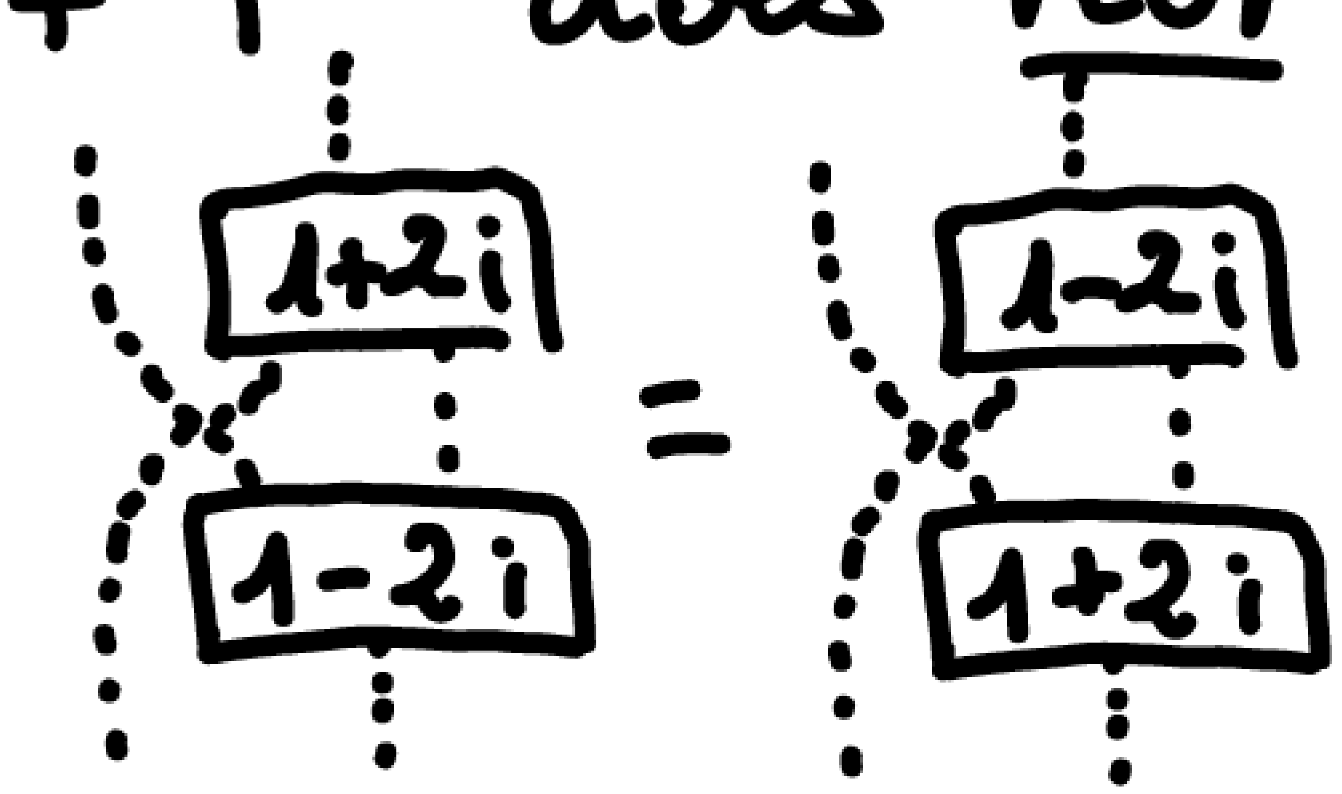
# Application to Quantum

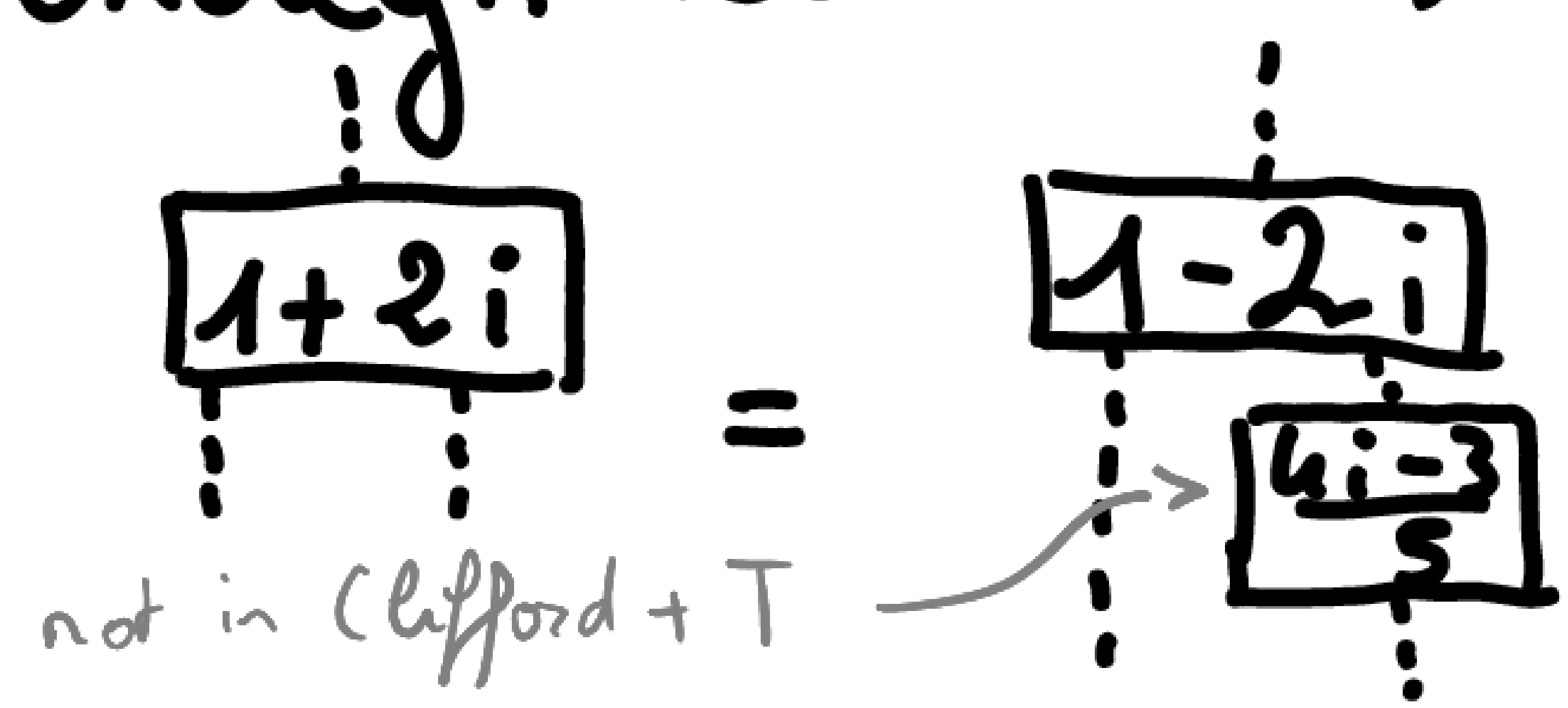
* FHilb has enough isometries
  ← cat. of linear maps between finite dim. Hilbert spaces

* Qubit has enough isometries
  ← cat. of linear maps between Hilbert spaces of the form $\mathbb{C}^{2^n}$

* Stab has enough isometries
  ← qubit linear maps composed of Clifford operators

* Clifford + T does <u>not</u> have enough isometries

E.g. 


but 

# What About Graphical Languages?

$$
\begin{tikzcd}
C_{iso} \arrow[r] \arrow[d, hook] \arrow[rr, bend left, "[\![\cdot]\!]_{iso}"] & C_{iso}^{!} \arrow[d] \arrow[rr, bend left] & D_{iso} \arrow[r] \arrow[d, hook] & D_{iso}^{!} \arrow[d] \\
C \arrow[r] \arrow[rr, bend right, "[\![\cdot]\!]"'] & C^{\pm} \arrow[rr, bend right, "[\![\cdot]\!]^{\pm}"'] & D \arrow[r] & D^{\pm}
\end{tikzcd}
$$

graphical language

interpretation

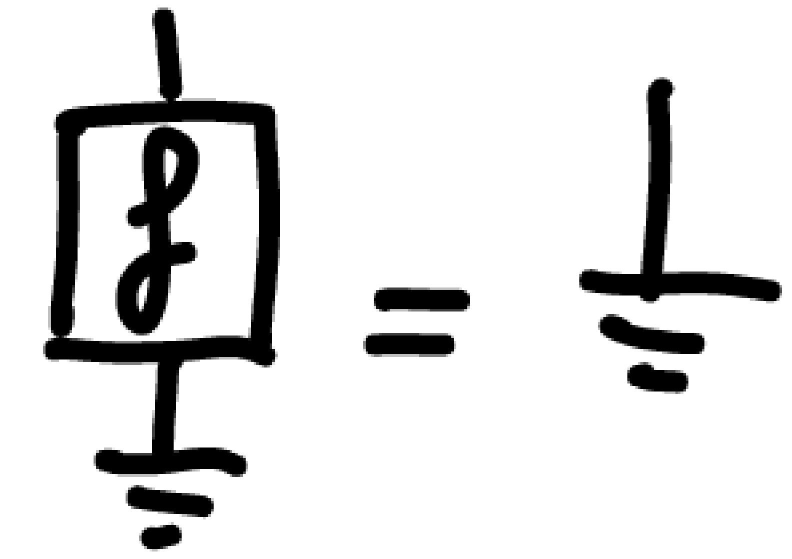
<u>Thm</u>: $[\![\cdot]\!]$ faithful $\wedge$ $[\![\cdot]\!]_{iso}$ surjective $\Rightarrow$ $[\![\cdot]\!]^{\pm}$ faithful

C is complete w.r.t. D

We can represent all isometries of D

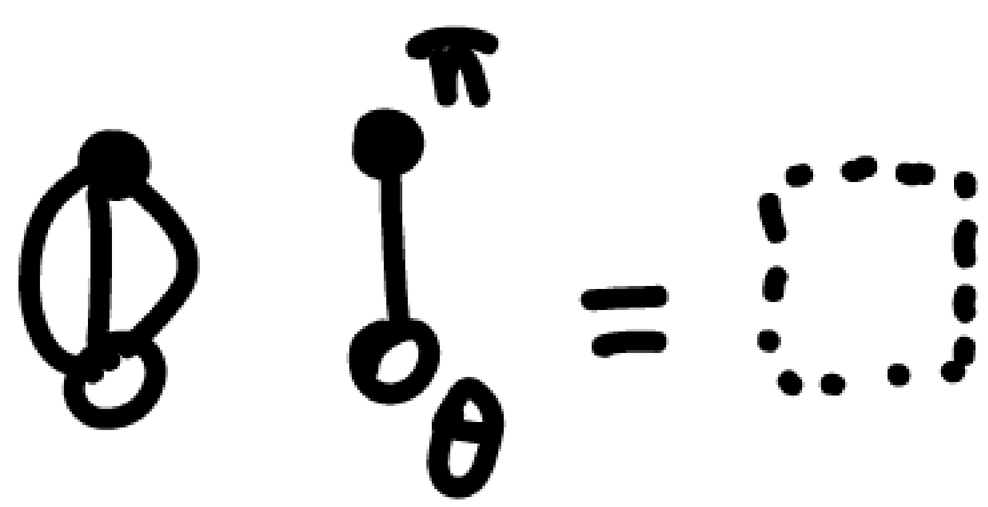
$C^{\pm}$ is complete w.r.t. $D^{\pm}$

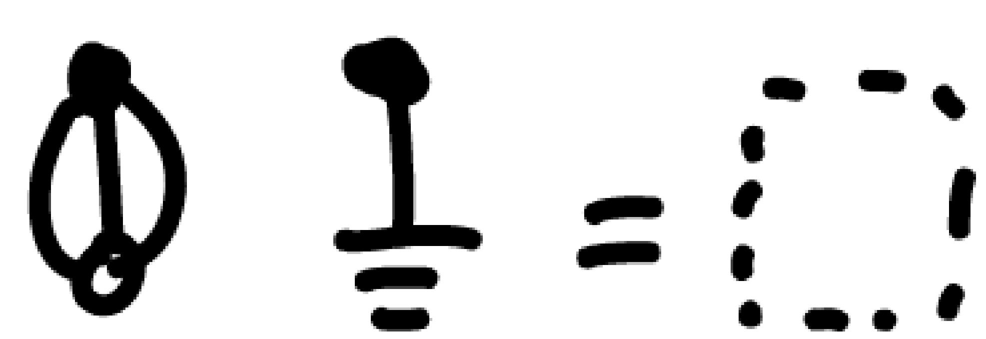
# From Infinitely Many to just a Few Axioms

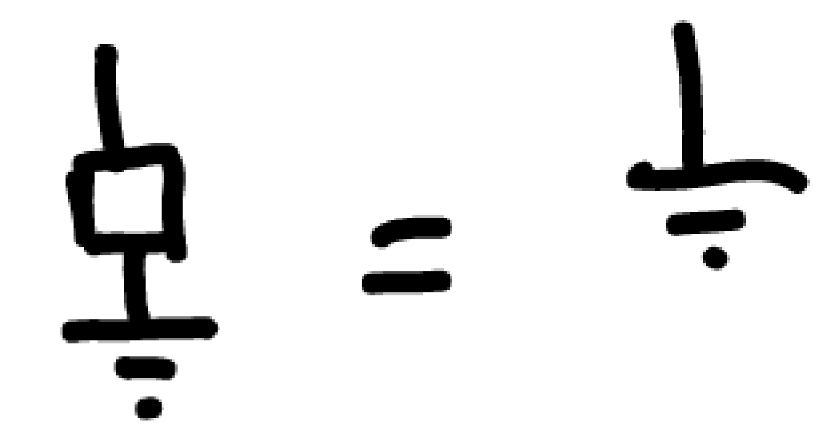
Eq. Th.$^{\text{⏚}}$ = Eq. Th. $\cup$ $\{$  / $f$ isometry $\}$

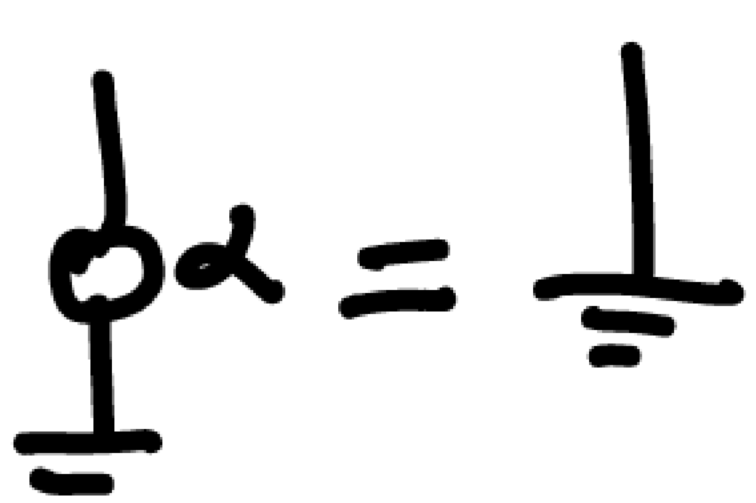
* Isometries generated by:

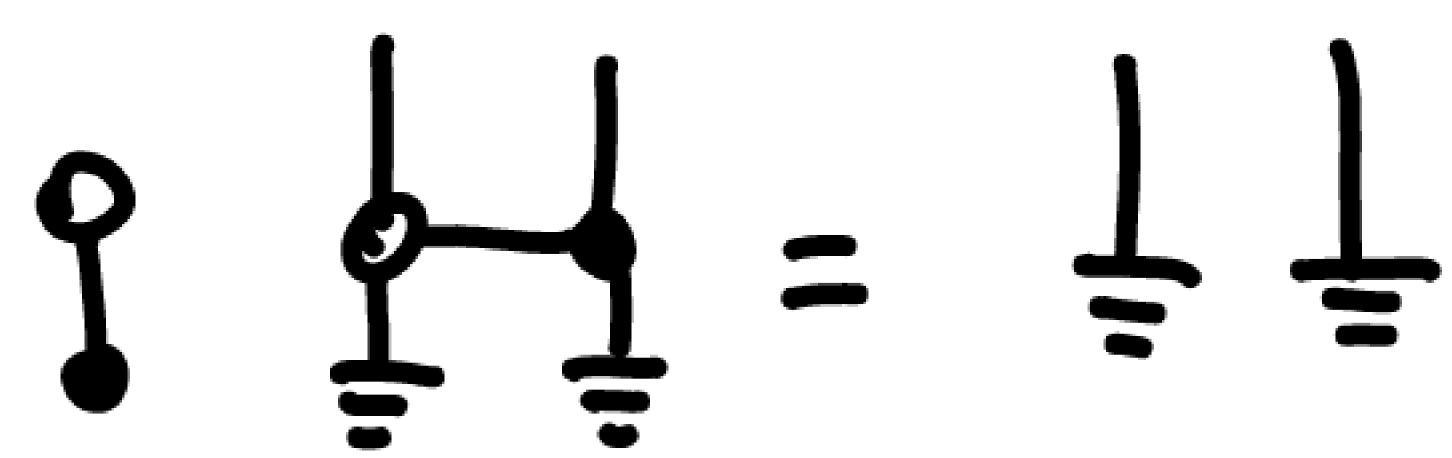
- $\langle e^{i\theta}, |0\rangle, H, P(\alpha), \mathrm{CNot} \rangle$ in **Qubit**
- $\langle e^{i\theta}, |0\rangle, H, P(\alpha), \mathrm{CNot} \rangle_{\theta \in \frac{\pi}{4}\mathbb{Z},\ \alpha \in \frac{\pi}{2}\mathbb{Z}}$ in Stab











# Representation of MBQC


no variables involved

* [Borgna, Perdrix, Valiron'21]

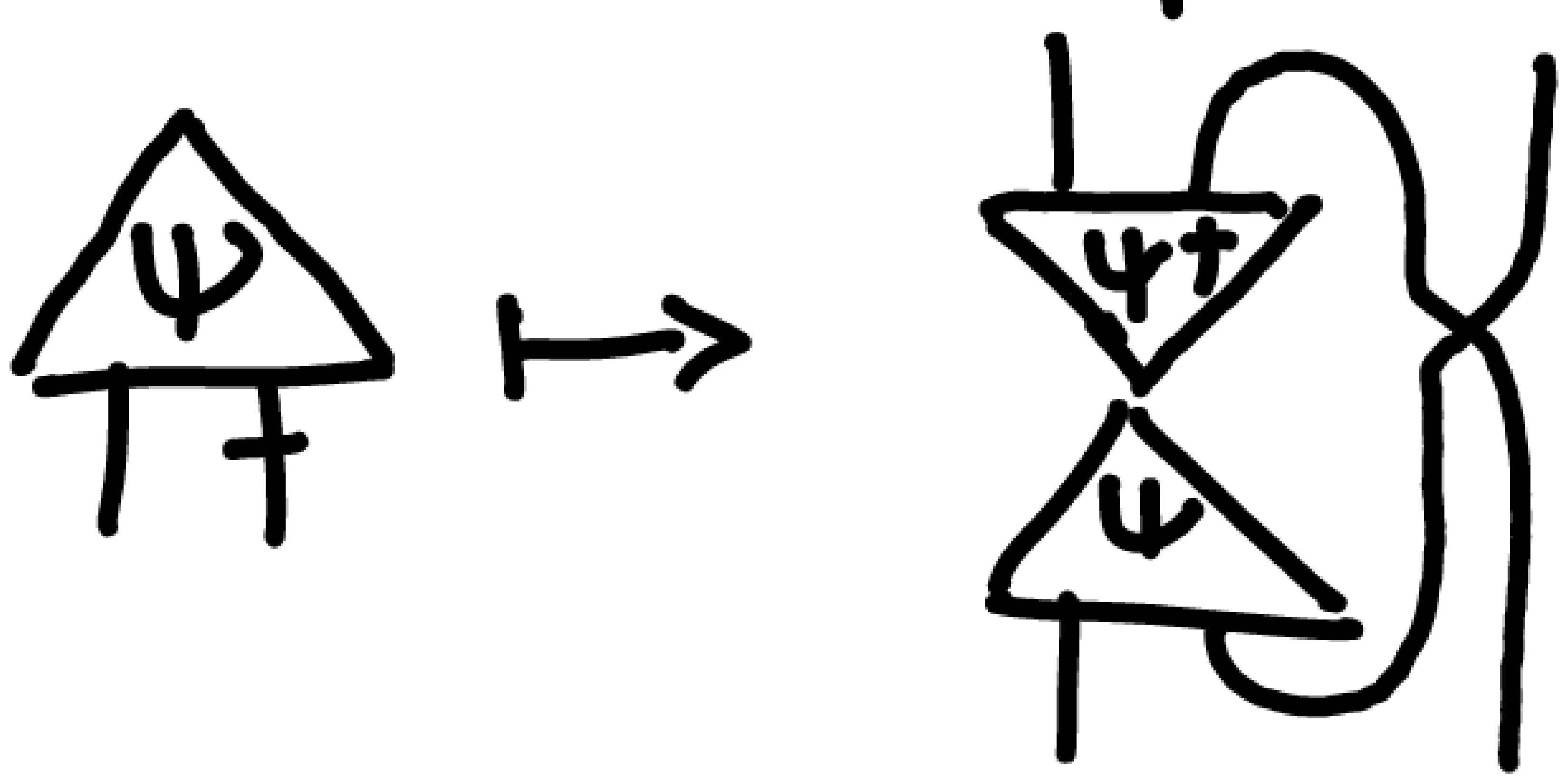
→ Simplification in $\mathrm{ZX}^{\text{⏚}}$

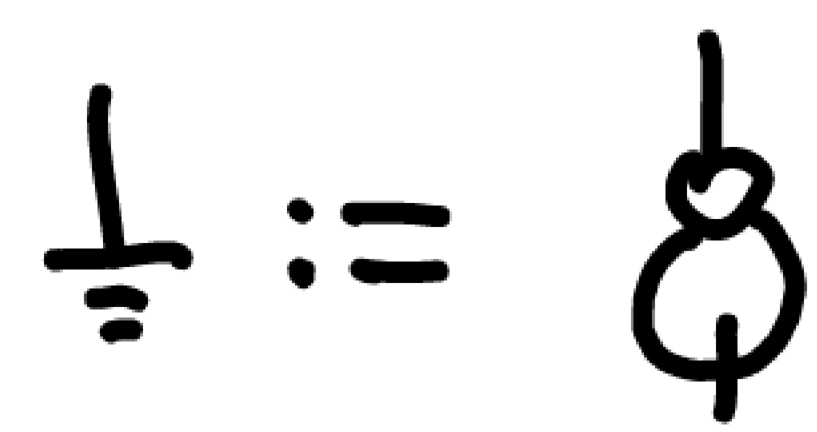
→ Identification of classical parts

# How to Deal with Clifford + T ?

* $\mathrm{ZX}_{\pi/4}^{\text{⏚}} \hookrightarrow \mathrm{ZX}^{\text{⏚}}$
prove equality using the full ZX.

* [Carette, Hoffreumon, Larroque, V'23]
~~partial trace~~ → partial transpose : $\dagger$





~~CPM~~ → Hermiticity-preserving maps
$\mathrm{ZX}_{\pi/4}^{\text{⏚}} \hookrightarrow \mathrm{ZX}_{\pi/4}^{\dagger}$

# Conclusion

* Discard construction
  - General
  - Universal properties

* Completenes in graphical languages if they have enough isometries.
  - 5 additional axioms for $ZX^{\ground}$ and $ZX^{\ground}_{\pi/2}$

* Clifford+T does not have enough isometries
  - $ZX^{\ground}_{\pi/4}$ not complete
  - Candidate: $ZX_{\pi/4}$ with partial transpose.
  - Simplifications still possible.