The circuit model, the one-way model, and the ZX-calculus

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Graphix workshop, 18 November 2024

Two models of quantum computation

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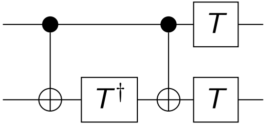
quantum circuit model



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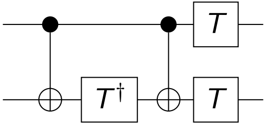

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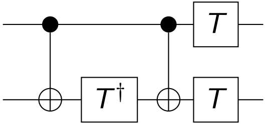

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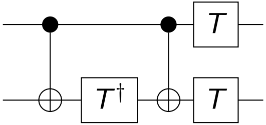

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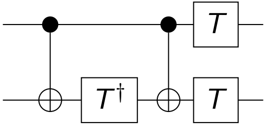

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[Raussendorf & Briegel 2001]

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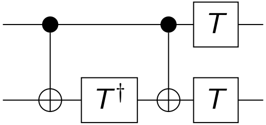

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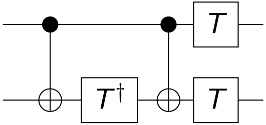

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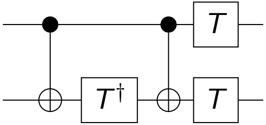

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- if goal is state preparation, need Pauli gates as correction at end

Motivation

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Question 2: What rewrite rules would be useful?

Outline

The one-way model of measurement-based quantum computing

A common formalism for circuits and MBQC

Translation and rewriting

Conclusions

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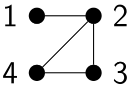
The resource states are **graph states** defined by simple graphs $G = (V, E)$:

- For each vertex in V, a qubit prepared in state $|+\rangle$.



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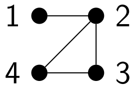

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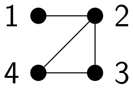

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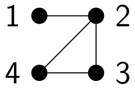

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All graph states are **stabiliser states**: eigenstates of certain tensor products of Pauli matrices.

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$$|+_{XY,\alpha}\rangle = \frac{1}{\sqrt{2}}(|0\rangle + e^{i\alpha}|1\rangle) \qquad |-_{XY,\alpha}\rangle = \frac{1}{\sqrt{2}}(|0\rangle - e^{i\alpha}|1\rangle) = Z|+_{XY,\alpha}\rangle$$

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Correction strategy: use **stabiliser property** of graph state to turn undesired outcome into desired one by applying Paulis to other qubits.

- Must have trivial effects on all qubits that are already measured.

Determinism and flow properties

Theorem [Browne et al. 2007, Mhalla et al. 2022]

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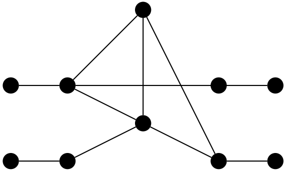

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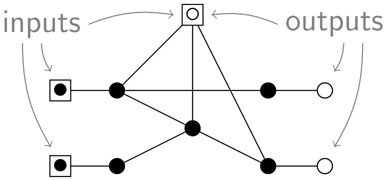

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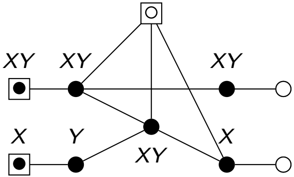

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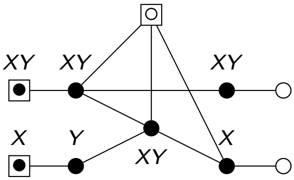

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- a partial order over the vertices,
- and a correction function,

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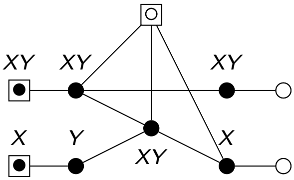

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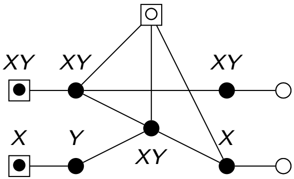

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Theorem [de Beaudrap 2008; Mhalla & Perdrix 2008; B. et al. 2021; Simmons 2021]

Flows can be found in polynomial time.

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An MBQC is represented in the graphical ZX-notation as follows:

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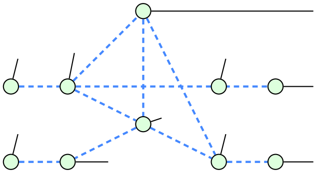

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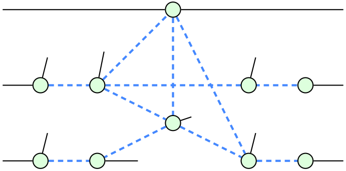

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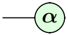
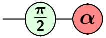
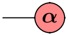
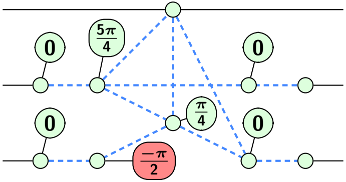
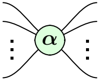
- additional vertices connected to some of the outgoing wires for the measurements: $XY \rightsquigarrow$ 

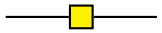
$XZ \rightsquigarrow$ 

$YZ \rightsquigarrow$ 



ZX generators and circuits in ZX-notation





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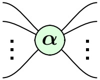

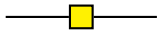

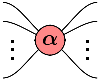

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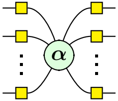

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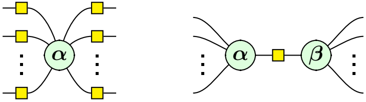

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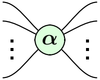

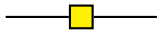

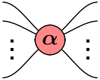

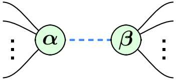

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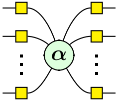
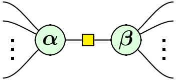
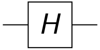




Gate 

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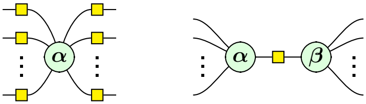

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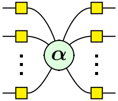

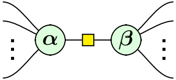

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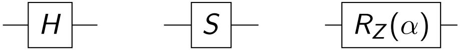

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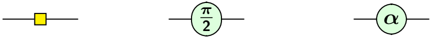
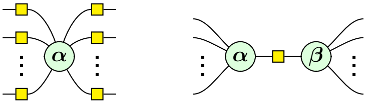
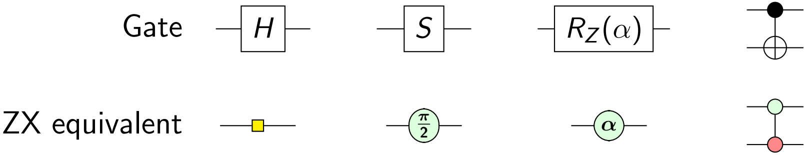


ZX generators and circuits in ZX-notation

$$\begin{pmatrix} 1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & e^{i\alpha} \end{pmatrix} \qquad \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$





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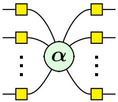
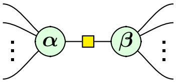
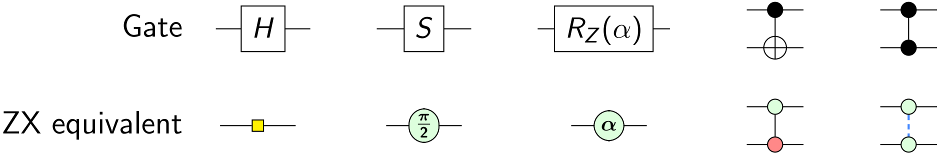
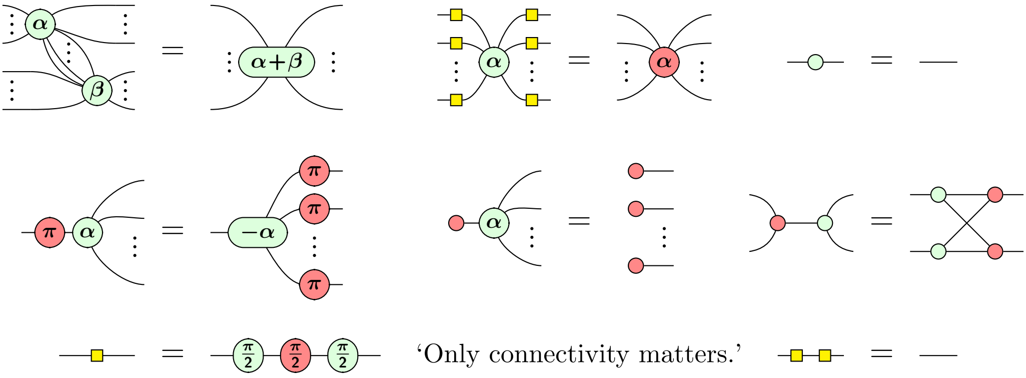






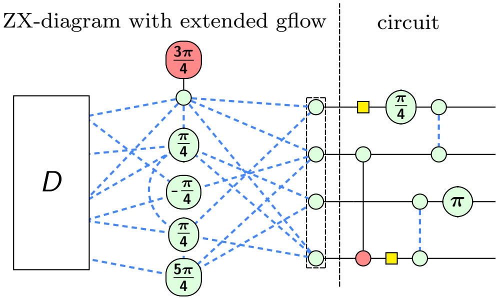
Sound and complete ZX-calculus rewrite rules (up to scalars)



This set is complete for the stabiliser ZX-calculus [B. 2014], can find overview over different complete rule sets in [van de Wetering 2021].

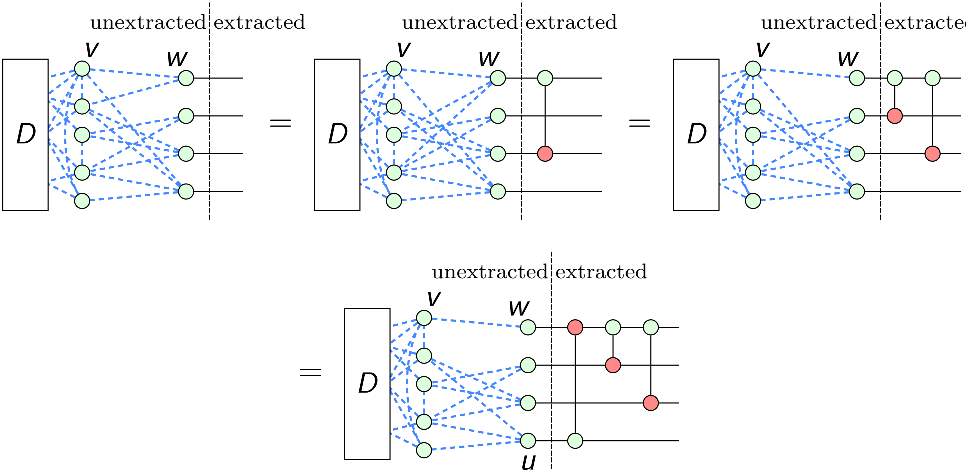
Outline

Ancilla-free circuit extraction: overview

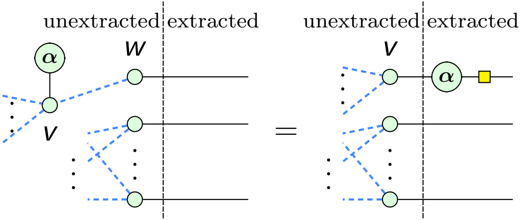


[Duncan et al. 2020; B., Miller-Bakewell, Felice, Lobski, van de Wetering 2021; Staudacher 2023]

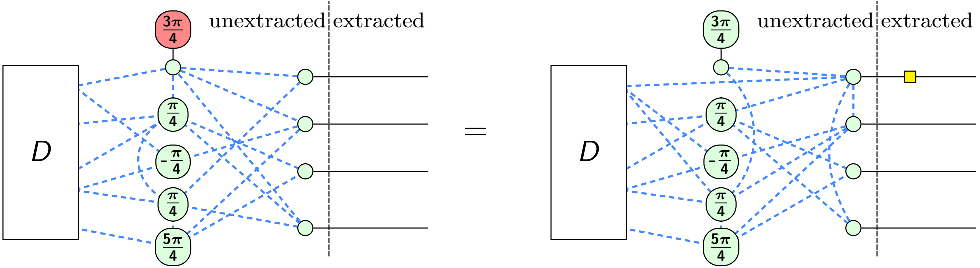
Simplify connections between frontier and unextracted layer



Extract a maximal vertex



If needed, change measurement type



Flow-preserving rewrite rules for the stabiliser ZX-calculus

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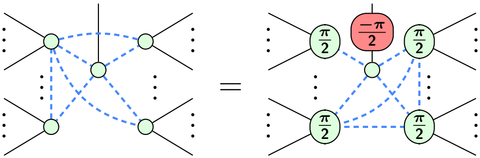
Local complementation



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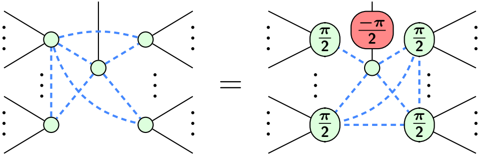

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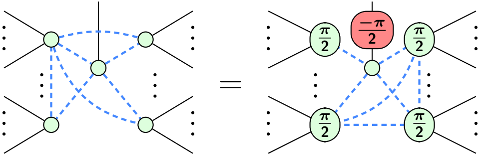
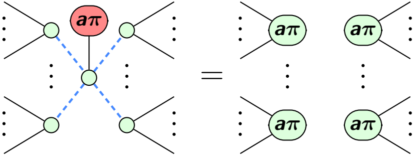


Z-deletion/insertion



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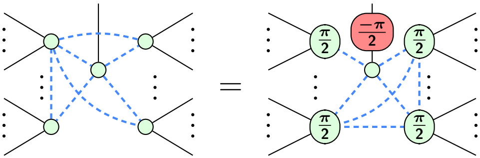

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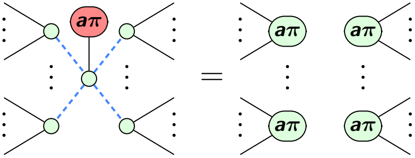

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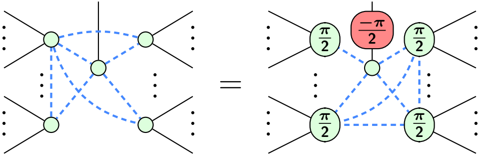

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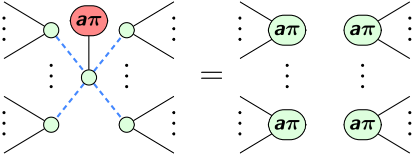

Theorem [McElvanney & B. 2023]

Suppose D and D' are two stabiliser ZX-diagrams with flow that both represent the same linear map. Then one can be rewritten into the other using local complementation, Z-insertion, and Z-deletion.

Further flow-preserving rewrite rules

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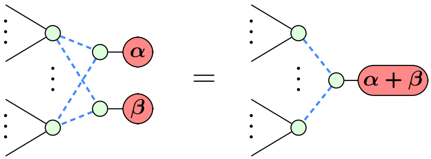
Phase gadget fusion/splitting



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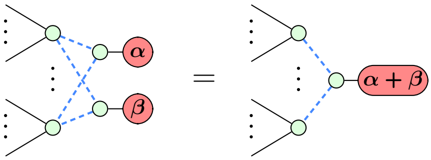

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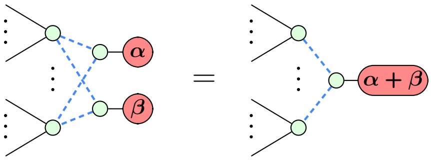
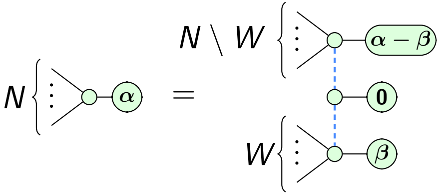


Vertex splitting/fusion



Applications of flow-preserving rewriting

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- T-count [Duncan et al. 2020]

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The one-way model of measurement-based quantum computing

A common formalism for circuits and MBQC

Translation and rewriting

Conclusions

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Thank you!